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Generalized Isotone Approximation and Related Topics

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1. INTRODUCTION

Recently Ubhaya [16–18] has studied the best weighted Chebyshev approximation of bounded functions defined on a partially ordered set by isotone functions [1]. This approximation has several interesting properties which distinguish it from the classical best weighted Chebyshev approximation by polynomials. The main properties of this type arise from explicit expressions for the error and for the set of all best weighted isotone approximations [16, 18].

In this paper we propose to study two other approximation problems which share many properties with the isotone approximation. More precisely, in Section 3 we consider the best weighted Chebyshev approximation by generalized isotone functions (cf. [1, p. 208; 9, Chapter XI]). Next, in Section 4 we present a numerical method for computing generalized isotone approximations. In particular, we note that our Theorems 3.2, 3.9 and 4.1 generalize Theorems 1 and 2 from [16] and Theorem 4 from [17], respectively. Moreover, in Theorem 3.3 we establish an explicit formula for a deviation of two subsets consisting of generalized isotone functions. In Section 5 we consider the best weighted Chebyshev problem which was studied by Timan in [14] (in fact Timan has dealt with the non-weighted Chebyshev approximation only). This section complements Timan's results by giving not only explicit formulae for the error of the best approximation and for the deviation of two subsets of approximants but also an explicit formula for the set of all best weighted approximations. At the same time we present simple proofs of these results. We note that Timan's approximation problem has many mathematical and physical applications [14]. In Section 5 we give another application of this problem. Namely, it will be useful in the proof of Theorem 5.6 which gives a formula for ball's measure of noncompactness [5, 6, 11, 12] in the space of continuous functions defined on a compact metric space with the weighted Chebyshev norm. This formula

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extends the result of Goldenštein and Markus [7] and gives the useful, lower and upper bounds for Kuratowski's measure of noncompactness [10]. We note that these expressions (bounds) for ball's (Kuratowski's) measure of noncompactness are needed, for example, to the verification of the assumptions in fixed point theorems for nonexpansive and condensing mappings (see e.g. [2, 3, 8, 12]). Finally, we note that in both approximation problems considered here explicit formulae are given for the error-determining set [4]. This is a new result even in the case of isotone approximation.

2. Preliminaries

Let X be a set, $B = B(X)$ be the linear space of all bounded real-valued functions defined on X, and $\|\cdot\|_w$ be the weighted Chebyshev norm on B

$$\|f\|_w = \sup_{x \in X} w(x)\,|f(x)|,$$

where w is a function in B such that

$$\delta = \inf_{x \in X} w(x) > 0. \tag{2.1}$$

If $w = 1$ on X, we shall write $\|\cdot\|$ instead of $\|\cdot\|_1$. For any $f \in B$ define

$$Z_f = \{x \in X: f(x) = 0\}$$

and

$$M_f = \{x \in X: w(x)\,|f(x)| = \|f\|_w\}.$$

Let G be a nonempty proper subset of B and

$$\theta = \theta_f = \inf_{h \in G} \|f - h\|_w. \tag{2.2}$$

If H is another proper subset of B, then its deviation $\theta(G, H)$ from G is defined as

$$\theta(G, H) = \sup_{g \in G} \inf_{h \in H} \|g - h\|_w. \tag{2.3}$$

An element $g \in G$ such that $\theta_f = \|f - g\|_w$ will be called the best weighted (Chebyshev) approximation to f in G. Denote by G_f and N_f the set of all best weighted approximations to f in G and the error-determining set [4]

$$N_f = \bigcap_{g \in G_f} M_{f-g}.$$

Let B be partially ordered in the usual way, i.e., let $g \leqslant h$ denote $g(x) \leqslant h(x)$ for all $x \in X$. For any $g, h \in B$ such that $g \leqslant h$ define

$$[g, h] = \{f \in B: g \leqslant f \leqslant h\}.$$

Let K be the set of all positive constant functions defined on X. For further considerations some results established in [13] should be mentioned.

DEFINITION 2.1 (see [13]). The subset G of B is called admissible with respect to the pair (f_1, f_2), $f_1, f_2 \in B$ if the following three conditions are satisfied:

(i) there exists $l \in G$, $l \geqslant f_1$, such that $g \geqslant l$ for every $g \in G$ such that $g \geqslant f_1$,

(ii) there exists $u \in G$, $u \leqslant f_2$, such that $g \leqslant u$ for every $g \in G$ such that $g \leqslant f_2$,

(iii) $g - \alpha \in G$ for every $\alpha \in K$ and $g \in G$ or $g + \alpha \in G$ for every $\alpha \in K$ and $g \in G$.

In particular, when $f_1 = f_2$ on X, we shall say that G is admissible with respect to f_1. Clearly, if G is admissible with respect to every $f \in F \subset B$ then G is admissible with respect to (f_1, f_2) for each $f_1, f_2 \in F$.

THEOREM 2.2 (see [13]). *Let a set G be admissible with respect to $(f_1, f_2) = (f - \theta/w,\ f + \theta/w)$, where $f \in B \backslash G$ and θ is given by* (2.2). *Suppose that l, u are defined in Definition* 2.1. *Then $l \leqslant u$, and the set G_f of all best weighted approximations is equal to $[l, u] \cap G$. In addition, if G is a convex set then the error-determining set N_f is equal to*

$$N_f = (Z_{f_1 - l} \cap Z_{f_1 - u}) \cup (Z_{f_2 - l} \cap Z_{f_2 - u}).$$

3. GENERALIZED ISOTONE APPROXIMATION

Throughout this section we shall assume that X is a partially ordered set with a partial order $\leqslant$. For $x \in X$ define the subsets L_x and U_x of X by

$$L_x = \{z \in X: z \leqslant x\} \qquad \text{and} \qquad U_x = \{z \in X: x \leqslant z\}.$$

Moreover, denote

$$T = \{(x, y) \in X \times Y: x \leqslant y\}.$$

Let $s \in B$ be such that

$$\tau = \inf_{x \in X} s(x) > 0. \tag{3.1}$$

Then define the set of generalized isotone functions P_s by

$$P_s = \left\{ g \in B : x \leqslant y \text{ implies } \begin{vmatrix} s(x), s(y) \\ g(x), g(y) \end{vmatrix} \geqslant 0 \right\}. \tag{3.2}$$

Clearly, P_s is a convex set and P_1 ($s = 1$ on X) coincides with the set of isotone functions. In the following, $[x, y]$ and (x, y) will always denote the closed interval in X and the pair in $X \times X$, respectively. For a fixed f in B define the following:

$$\begin{aligned}
d(x, y) &= \frac{w(x)\, w(y)}{w(x)\, s(x) + w(y)\, s(y)} \, |s(y) f(x) - s(x) f(y)|, \\
m(x, y) &= \frac{w(x) f(x) + w(y) f(y)}{w(x)\, s(x) + w(y)\, s(y)}, \\
T_d &= \{(x, y) \in T : d(x, y) = \theta\}, \qquad (3.3) \\
T_0 &= \bigcup_{(x, y) \in T_d} [x, y], \\
Q &= \bigcup_{(x, y) \in T_d} \{x, y\} \qquad \text{and} \qquad \theta = \theta_f = \inf_{g \in P_s} \|f - g\|_w.
\end{aligned}$$

LEMMA 3.1. *For every $r \in B$ the set P_s is admissible with respect to r. Moreover, l and u from Definition 2.1 are equal to*

$$l(x) = s(x) \sup_{z \in L_x} r(x)/s(z)$$

and

$$u(x) = s(x) \inf_{z \in U_x} r(z)/s(z)$$

for all $x \in X$.

Proof. From the definitions of l, u and s it immediately follows that l, $u \in B$, $l \geqslant r$ and $u \leqslant r$. Moreover, if $\varepsilon > 0$ and $x, y \in X$ $(x \leqslant y)$ are arbitrarily fixed then there exist $t \in L_x$ and $v \in U_y$ such that

$$l(x) \leqslant s(x)\, r(t)/s(t) + \varepsilon \qquad \text{and} \qquad u(y) \geqslant s(y)\, r(v)/s(v) - \varepsilon.$$

Hence

$$s(y)\,l(x) \leqslant s(x)\,s(y)\,r(t)/s(t) + \varepsilon s(y) \leqslant s(x)\,l(y) + \varepsilon\,\|s\|$$

and

$$s(x)\,u(y) \geqslant s(y)\,s(x)\,r(v)/s(v) - \varepsilon s(x) \geqslant s(y)\,u(x) - \varepsilon\,\|s\|.$$

Since ε is arbitrary, then from the last two inequalities it follows that $l, u \in P_s$. Now, let $x \in X$ and $g \in P_s$ such that $g \geqslant r$ be arbitrarily fixed. Then $s(z)\,g(x) \geqslant s(x)\,g(z) \geqslant s(x)\,r(z)$ for every $z \in L_x$. Hence $g(x) \geqslant l(x)$. This establishes (i) in Definition 2.1. On the other hand, if $g \in P_s$ is such that $g \leqslant r$ then $s(z)\,g(x) \leqslant s(x)\,g(z) \leqslant s(x)\,r(z)$ for every $z \in U_x$ and so $g(x) \leqslant u(x)$. Hence $g \leqslant u$, which proves (ii) in Definition 2.1. Since condition (iii) in Definition 2.1 is obvious for P_s, the lemma is proved. ∎

THEOREM 3.2. *Let $f \in B \backslash P_s$, and θ, l and u be defined by $\theta = \sup_{(x,y) \in T} d(x, y)$, $l(x) = s(x) \sup_{z \in L_x} f_1(z)/s(z)$ and $u(x) = s(x) \inf_{z \in U_x} f_2(z)/s(z)$, where $f_1 = f - \theta/w$, $f_2 = f + \theta/w$, and d is defined in (3.3). Then $l, u \in P_s$, the set of all best weighted approximations to f in P_s is equal to $[l, u] \cap P_s$ and the error $\theta_f = \inf_{g \in P_s} \|f - g\|_w$ is equal to θ. Moreover, the error-determining set N_f is equal to*

$$N_f = (Z_{f_1 - l} \cap Z_{f_1 - u}) \cup (Z_{f_2 - u} \cap Z_{f_2 - l}).$$

Proof. By Theorem 2.2 and Lemma 3.1 it is sufficient to prove that θ_f is equal to θ. Let us suppose that $g \in P_s$ is arbitrarily fixed. Then $s(x)\,g(y) - s(y)\,g(x) \geqslant 0$ for all $(x, y) \in T$. Hence

$$\begin{aligned} s(y)\,f(x) - s(x)\,f(y) &\leqslant s(y)\,f(x) - s(x)\,f(y) + s(x)\,g(y) - s(y)\,g(x) \\ &\leqslant [s(y)/w(x) + s(x)/w(y)]\,\|f - g\|_w \end{aligned}$$

for all $(x, y) \in T$. Consequently, we have $\|f - g\|_w \geqslant d(x, y)$ for all $g \in P_s$ and $(x, y) \in T$. This implies that $\theta_f \geqslant \theta$. Hence the proof will be complete if we show that $\|f - l\|_w \leqslant \theta$ for $l \in P_s$ defined above. At first, we note that

$$w(x)[f(x) - l(x)] \leqslant w(x)[f(x) - f_1(x)] = \theta \tag{3.4}$$

for all $x \in X$. Secondly, in view of the definition of l, it follows that for every $\varepsilon > 0$ there exists $t \in L_x$ such that

$$l(x) \leqslant s(x)\,f_1(t)/s(t) + \varepsilon.$$

Hence by the definitions of f_1 and θ we obtain

$$\begin{aligned}
w(x)[l(x)-f(x)] &\leqslant w(x)[s(x)\,f_1(t)/s(t)-f(x)+\varepsilon] \\
&= w(x)\left[s(x)\,f(t)/s(t)-f(x)-\frac{\theta s(x)}{w(t)\,s(t)}+\varepsilon\right] \\
&\leqslant w(x)\Big\{s(x)\,f(t)/s(t)-f(x) \\
&\quad -\frac{w(x)\,s(x)}{w(t)\,s(t)+w(x)\,s(x)}\,[s(x)\,f(t)/s(t)-f(x)]+\varepsilon\Big\} \\
&= d(t,x)+\varepsilon w(x)\leqslant \theta+\varepsilon\,\|w\|.
\end{aligned}$$

Since ε and x are arbitrary, then $w(x)[l(x)-f(x)]\leqslant\theta$ for all $x\in X$. This inequality and (3.4) imply that $\|f-l\|_w\leqslant\theta$. ∎

Now, let $r\in B$ be such that

$$\inf_{x\in X} r(x)>0.$$

Denote

$$P_r^0=\{f\in P_r : 0\leqslant f\leqslant r\}.$$

Then, we have

THEOREM 3.3. *The deviation of P_s from P_r^0 is equal*

$$\theta(P_r^0,P_s)=\sup_{(x,y)\in T} e(x,y),$$

where

$$e(x,y)=\frac{w(x)\,w(y)}{s(x)\,w(y)+s(y)\,w(y)}\,[s(y)\,r(x)-s(x)\,r(y)].$$

Proof. From Theorem 3.2 and (2.3) we obtain

$$\theta(P_r^0,P_s)=\sup_{s\in P_r^0}\ \sup_{(x,y)\in T} d(x,y), \tag{3.5}$$

where d is defined in (3.3). Hence by virtue of $r\in P_r^0$ we have

$$\theta(P_r^0,P_s)\geqslant \sup_{(x,y)\in T} e(x,y). \tag{3.6}$$

On the other hand, let $f \in P_r^0$ be arbitrarily fixed, i.e., f be such that $0 \leqslant f \leqslant r$ and $f(y) \geqslant f(x)\, r(y)/r(x)$ for all $(x, y) \in T$. Then

$$s(y)\, f(x) - s(x)\, f(y) \leqslant s(y)\, f(x) - s(x)\, f(x)\, r(y)/r(x)$$
$$= \frac{f(x)}{r(x)}\, [s(y)\, r(x) - s(x)\, r(y)]$$

for all $(x, y) \in T$. Hence

$$d(x, y) \leqslant \frac{f(x)}{r(x)} \sup_{(x,y) \in T} e(x, y) \leqslant \sup_{(x,y) \in T} e(x, y)$$

for all $(x, y) \in T$. Taking the supremum of the left-hand side of the last inequality over $(x, y) \in T$ and $f \in P_r^0$ and using of (3.5), we obtain

$$\theta(P_r^0, P_s) \leqslant \sup_{(x,y) \in T} e(x, y).$$

This inequality together with (3.6) give the result. ∎

Before we state our next theorem concerning other properties of generalized isotone weighted approximations, we shall establish four auxiliary lemmas. Some of these lemmas seem to be independently interesting.

LEMMA 3.4. *For each pair* $(x, y) \in T_d$ *we have*

$$x \in Z_{f_1 - l} \cap Z_{f_1 - u} \qquad \text{and} \qquad y \in Z_{f_2 - l} \cap Z_{f_2 - u}.$$

Proof. Let us suppose that $(x, y) \in T_d$, i.e., that $x \leqslant y$ and $\theta = d(x, y)$. This equality is equivalent to $f_1(x) = s(x)\, f_2(y)/s(y)$. By noting that $f_1 \leqslant l \leqslant u \leqslant f_2$ and $x \leqslant y$ we conclude that

$$f_1(x) = s(x)\, f_2(y)/s(y) \geqslant u(x) \geqslant l(x) \geqslant f_1(x)$$

and

$$f_2(y) = s(y)\, f_1(x)/s(x) \leqslant l(y) \leqslant u(y) \leqslant f_2(y).$$

Hence $l(x) = u(x) = f_1(x)$ and $l(y) = u(y) = f_2(y)$. ∎

By Theorem 3.2 and Lemma 3.4 we obtain the following:

COROLLARY 3.5. $Q \subset N_f$.

LEMMA 3.6. *Let $(x, y) \in T_d$. Then we have*

$$s(t)\, l(x) = s(x)\, l(t) = s(t)\, s(x)\, m(x, y) = s(x)\, u(t) = s(t)\, u(x)$$

for all $t \in [x, y]$. In particular, $[x, y] \subset Z_{l-u}$ and $T_0 \subset Z_{l-u}$.

Proof. Let $t \in [x, y]$, where $(x, y) \in T_d$. Then from Lemma 3.4 and from the definition of the set T_d we obtain

$$\begin{aligned} l(x) &= u(x) = f(x) = f(x) - \theta/w(x) \\ &= f(x) - \frac{w(y)}{w(x)\, s(x) + w(y)\, s(y)} [s(y)\, f(x) - s(x)\, f(y)] = s(x)\, m(x, y) \end{aligned}$$

and

$$l(y) = u(y) = f_2(y) = f(y) + \theta/w(y) = s(y)\, m(x, y).$$

Hence by the fact that $l, u \in P_s$ we have

$$\begin{aligned} s(t)\, l(x) &\leqslant s(x)\, l(t) \leqslant s(x)\, s(t)\, l(y)/s(y) \\ &= s(t)\, s(x)\, m(x, y) = s(t)\, l(x) \end{aligned}$$

and

$$\begin{aligned} s(t)\, u(x) &\leqslant s(x)\, u(t) \leqslant s(x)\, s(t)\, u(y)/s(y) = s(x)\, s(t)\, m(x, y) \\ &= s(t)\, u(x) = s(t)\, l(x). \end{aligned}$$

From these the lemma follows at once. ∎

Now, let X be a chain. In the following we shall assume that X is endowed with the interval (intrinsic) topology. It is well known [1] that a chain X is a normal Hausdorff space under its intrinsic topology. Denote by $C_b = C_b(X)$ the subspace of B consisting of all continuous functions on a chain X. In particular, if X is a compact chain, we shall write C instead C_b.

LEMMA 3.7. *Let $w, s, f \in C_b$. Then $l, u \in C_b$.*

Proof. We show that $l \in C_b$; the proof for u is similar and so is omitted here. Obviously, the function l, by the assumptions about w, s and f, is bounded on X. From the definition of l we may write

$$l(y) = \max \left\{ s(y)\, l(x)/s(x),\ s(y) \sup_{z \in [x, y]} f_1(z)/s(z) \right\}$$

for any $x, y \in X$ such that $x \leqslant y$. Hence by the fact that $l \in P_s$ and $l \geqslant f_1$ we have

$$\begin{aligned} 0 &\leqslant s(x)\, l(y) - s(y)\, l(x) \\ &= \max \left\{ 0, s(x)\, s(y) \sup_{z \in [x,y]} f_1(z)/s(z) - s(y)\, l(x) \right\} \\ &\leqslant \max \left\{ 0, s(x)\, s(y) \left[\sup_{z \in [x,y]} f_1(z)/s(z) - f_1(x)/s(x) \right] \right\} \end{aligned} \tag{3.7}$$

for all $x, y \in X$, $x \leqslant y$. Now, let $\varepsilon > 0$ and $x \in X$ be arbitrarily fixed. We assume that $l \not\equiv 0$ on X, since otherwise the proof is trivial. From the continuity of the functions s and f_1/s on X (see (3.1)) it follows that there exists the open interval 0_x in X containing x such that

$$|s(x) - s(y)| < \tfrac{1}{4}\varepsilon/\|l\| \qquad \text{and} \qquad |f_1(x)/s(x) - f_1(y)/s(y)| < \tfrac{1}{2}\varepsilon/\|s\|^2$$

for all $y \in 0_x$. From this and from (3.7) we conclude that

$$\begin{aligned} &|s(y)\, l(y) - s(x)\, l(x)| \\ &\quad = |[l(x) + l(y)][s(y) - s(x)] + [s(x)\, l(y) - s(y)\, l(x)]| \\ &\quad \leqslant (|l(x)| + |l(y)|)\, |s(y) - s(x)| + |s(x)\, l(y) - s(y)\, l(x)| \\ &\quad < \varepsilon/2 + \varepsilon/2 = \varepsilon \end{aligned}$$

for all $y \in 0_x$. Hence $l \cdot s \in C_b$. Consequently, by (3.1) we conclude that $l \in C_b$. ∎

LEMMA 3.8. *Let X be a compact chain, $w, s \in C = C(X)$ and $f \in C \backslash P_s$. Then $Z_{l-u} = T_0 \neq \varnothing$. Moreover, there exist a positive integer k and a sequence of pairs $\{(x_i, y_i)\}_{i=1}^{k}$ such that $(x_i, y_i) \in T_d$, $x_1 < y_1 < x_2 < y_2 < \cdots < x_k < y_k$ and*

$$T_0 = \bigcup_{i=1}^{k} [x_i, y_i].$$

Proof. Let $x \in Z_{l-u}$ be arbitrarily fixed. By compactness of L_x and U_x and a continuity of f_1/s and f_2/s it follows that there exist $t_1 \in L_x$ and $t_2 \in U_x$ such that

$$l(x) = f_1(t_1)\, s(x)/s(t_1) = f_2(t_2)\, s(x)/s(t_2) = u(x).$$

Hence

$$[f(t_1) - \theta/w(t_1)]\, s(t_2) = [f(t_2) + \theta/w(t_2)]\, s(t_1)$$

and so

$$\theta = d(t_1, t_2).$$

Thus $(t_1, t_2) \in T_d$ and $x \in [t_1, t_2] \subset T_0$. This and Lemma 3.6 imply that $Z_{l-u} = T_0$. Now, let us suppose, contradicting the lemma, that there exist $(x_i, y_i) \in T_d$ $(i = 1, 2, \ldots)$ such that $x_1 < y_1 < x_2 < y_2 < \cdots$. Then by Lemma 3.4 and the fact that $\theta > 0$ (since $f \notin P_s$) we have

$$f(x_i) = l(x_i) = \theta/w(x_i) \geqslant \theta/\|w\| > 0$$

and

$$f(y_i) - l(y_i) = -\theta/w(y_i) \leqslant -\theta/\|w\| < 0$$

for all i. Hence by the compactness of X we obtain a contradiction with a continuity of $f - l$ (see Lemma 3.7). ∎

From Theorem 3.2, Corollary 3.5 and Lemmas 3.6, 3.7 and 3.8 we obtain

THEOREM 3.9. *Let X be a compact chain and let $w, s \in C = C(X)$ and $f \in C \backslash P_s$. Then, in addition to the results of Theorem* 3.2, *$l, u \in P_s \cap C$ and*

$$\theta_f = \inf_{g \in P_s} \|f - g\|_w = \inf_{g \in P_s \cap C} \|f - g\|_w = \|f - l\|_w = \|f - u\|_w.$$

Moreover, $Q \subset N_f$ and $Z_{l-u} = T_0$. The set Z_{l-u} can be given in the form

$$Z_{l-u} = \bigcup_{i=1}^{k} [x_i, y_i].$$

where k is some positive integer, $x_1 < y_1 < x_2 < y_2 < \cdots < x_k < y_k$ and $(x_i, y_i) \in T_d$ for all i. Also, we have

$$l(x) = u(x) = s(x)\, m(x_i, y_i) \qquad \textit{for all } x \in [x_i, y_i]\ (i = 1, 2, \ldots, k),$$

where

$$s(x_{i-1})\, m(x_{i-1}, y_{i-1}) < s(x_i)\, m(x_i\, y_i) \qquad (i = 2, 3, \ldots, k).$$

Finally, note that by setting $X = [a, b]$ and $s \equiv 1$ in our Theorems 3.2 and 3.9 we obtain Theorems 1 and 2 from Ubhaya's paper [16], respectively.

4. Numerical Methods

In this section we will restrict our attention to the numerical determination of the generalized isotone weighted approximations l and u and its error θ in the space $C(X)$ with $X=[a,b]$. Numerical methods presented here generalized Ubhaya's algorithms for isotone approximations given in [17]. In these methods we replace the interval X by a finite set of points and seek the generalized isotone weighted approximations l and u which are best on that set. Note, that from the formulae for l and u given in Theorem 3.2 it follows that l and u can be determined exactly by a computer in the case of discrete set X.

It is necessary to establish some notations. Let us denote by $\omega(h;\lambda)$, $0 \leqslant \lambda \leqslant b-a$, the modulus of continuity [15] of $h \in C(X)$, $X=[a,b]$, i.e., let $\omega(h;0)=0$ and

$$\omega(h;\lambda)=\max_{|x-y|\leqslant\lambda}|h(x)-h(y)|, \qquad 0<\lambda\leqslant b-a.$$

Suppose that the function g belongs to $P_s(X_k)\cap C(X_k)$ and $s \in C(X)$, where $X_k=\{x_i\}_{i=0}^k$ $(a=x_0<x_1<\cdots<x_k=b)$. Then the function g, defined by

$$\begin{aligned} g(x)&=s(x)[a_i+b_i x], \qquad x\in[x_{i-1},x_i],\\ a_i&=\frac{x_i\,g(x_{i-1})/s(x_{i-1})-x_{i-1}\,g(x_i)/s(x_i)}{x_i-x_{i-1}},\\ b_i&=\frac{g(x_i)/s(x_i)-g(x_{i-1})/s(x_{i-1})}{x_i-x_{i-1}} \qquad (i=1,2,\ldots,k), \end{aligned} \tag{4.1}$$

indeed belongs to $P_s(X)\cap C(X)$ by (3.1), $g\in C(X)$. Since $g\in P_s(X_k)$ then $s(x_{i-1})\,g(x_i)-s(x_i)\,g(x_{i-1})\geqslant 0$ and so $b_i\geqslant 0$ for all i. Hence $s(x)\,g(y)-s(y)\,g(x)=b_i s(x)\,s(y)(y-x)\geqslant 0$ for all x, y $(x_{i-1}\leqslant x\leqslant y\leqslant x_i)$, i.e., $g\in P_s(X)$.

In the following we shall use also notations from Section 3. Now, we state numerical methods of the determination of l, u and θ, and show their convergence and rates of convergence of various quantities involved.

Theorem 4.1. *Let $X=[a,b], f\in C(X)\backslash P_s$ and $w, s\in C(X)$. Let $X_n\subset X$ be a sequence of finite sets such that $a, b\in X_n$ $(n=1,2,\ldots)$ and*

$$\rho_n=\sup_{x\in X}\inf_{x\in X_n}|x-y|\to 0, \qquad as \quad n\to\infty. \tag{4.2}$$

Denote

$$\begin{aligned}
\theta_n &= \sup_{(x,y)\in T_n} d(x,y),\\
l_n(x) &= s(x) \sup_{z\in L_{x,n}} f_{1,n}(z)/s(z),\\
u_n(x) &= s(x) \inf_{z\in U_{x,n}} f_{2,n}(z)/s(z),\\
\tilde{l}_n(x) &= s(x) \sup_{z\in L_{x,n}} f_1(z)/s(z),\\
\tilde{u}_n(x) &= s(x) \inf_{z\in U_{x,n}} f_2(z)/s(z),
\end{aligned} \tag{4.3}$$

where

$$x\in X_n, \qquad T_n = T\cap (X_n\times X_n), \qquad L_{x,n} = L_x\cap X_n,$$
$$U_{x,n} = U_x\cap X_n, \qquad f_{1,n} = f-\theta_n/w, \qquad f_{2,n} = f+\theta_n/w.$$

Additionally, let l_n, u_n, $\tilde{l}_n$ *and* $\tilde{u}_n$ *be expanded on* X *according to* (4.1). *Then we have*

(i) $\theta_n \leqslant \theta$ *for all* n *and* $\theta_n \to \theta$ *as* $n\to\infty$ *according to*

$$0\leqslant \theta-\theta_n \leqslant c_1\omega(f;\rho_n)+c_2\omega(w;\rho_n)+c_3\omega(s;\rho_n)$$

where

$$c_1 = \frac{\|w\|^2\,\|s\|}{\tau\delta}, \qquad c_2 = \frac{\|w\|^2\,\|s\|^2\,\|f\|}{\tau^2\delta^2},$$
$$c_3 = \frac{\|w\|^2\,\|f\|}{\tau\delta}\left(1+\frac{\|w\|\,\|s\|}{\tau\delta}\right),$$

and τ, δ *are defined in* (2.1) *and* (3.1), *respectively*;

(ii) g_n $(=l_n$ *or* $u_n)$ *converges uniformly to* g $(=l$ *or* u, *respectively*) *according to*

$$\|g_n-g\|\leqslant c_4\omega(f;\rho_n)+c_5\omega(w;\rho_n)+c_6\omega(s;\rho_n)$$

where

$$c_4 = \frac{\|s\|}{\tau}\left(\frac{2\,\|s\|}{\tau}+c_1/\delta\right), \qquad c_5 = \frac{\|s\|}{\tau\delta}\left(c_2+\frac{2\theta\,\|s\|}{\tau\delta}\right),$$
$$c_6 = \frac{\|s\|}{\tau}\left(2\,\|f\|/\tau+c_3/\delta+\frac{2\theta\,\|w\|}{\tau\delta^2}\right);$$

(iii) $\tilde{l}_n \leqslant l \leqslant u \leqslant \tilde{u}_n$ *and* $\tilde{g}_n$ $(=\tilde{l}_n$ *or* $\tilde{u}_n)$ *converges uniformly to* g $(=l$ *or* u, *respectively*) *according to*

$$\|\tilde{g}_n - g\| \leqslant \frac{2\|s\|^2}{\tau^2}\,\omega(f;\rho_n) + \frac{2\theta\|s\|^2}{\tau\delta^2}\,\omega(w;\rho_n) + \frac{2\|s\|}{\tau^2}\left(\|f\| + \frac{\theta\|w\|}{\delta^2}\right)\omega(s;\rho_n);$$

(iv) *If* $X_n \subset X_{n+m}$ $(n, m \geqslant 1)$ *then*

$$g_n(x) \leqslant g_{n+m}(x) \leqslant h_{n+m}(x) \leqslant h_n(x)$$

for all $x \in X_n$, *where* $g_k = l_k$ (*or* $\tilde{l}_k$) *and* $h_k = u_k$ (*or* $\tilde{u}_k$, *respectively*) *for* $n = n, n+m$.

Proof. (i) From the compactness of X and the continuity of all considered functions it follows that there exist $x, y \in X$ $(x < y)$ such that $\theta = d(x, y)$. Since X_n is a finite set, then by (4.2) there exists $(x_n, y_n) \in T_n$ such that $|x_n - x| \leqslant \rho_n$ and $|y_n - y| \leqslant \rho_n$. By the definitions of θ_n and d we then have

$$\begin{aligned}
0 &\leqslant \theta - \theta_n \leqslant d(x, y) - d(x_n, y_n)\\
&= \frac{w(x)\,w(y)}{s(w)\,w(x) + s(y)\,w(y)}\,\{s(y)[f(x) - f(x_n)] + s(x)[f(y_n) - f(y)]\\
&\quad + f(x_n)[s(y) - s(y_n)] + f(y_n)[s(x_n) - s(x)]\}\\
&\quad + \frac{s(y_n)\,f(x_n) - s(x_n)\,f(y_n)}{[s(x)\,w(x) + s(y)\,w(y)][s(x_n)\,w(x_n) + s(y_n)\,w(y_n)]}\\
&\quad \times \{w(x)\,w(x_n)\,s(x_n)[w(y) - w(y_n)] + w(x)\,w(x_n)\,w(y_n)[s(x_n) - s(x)]\\
&\quad + w(y)\,w(y_n)\,s(y_n)[w(x) - w(x_n)] + w(y)\,w(y_n)\,w(x_n)[s(y_n) - s(y)]\}\\
&\leqslant \frac{\|w\|^2}{\tau\delta}\,[\|s\|\,\omega(f;\rho_n) + \|f\|\,\omega(s;\rho_n)]\\
&\quad + \frac{\|s\|\,\|f\|}{\tau^2\delta^2}\,[\|w\|^2\,\|s\|\,\omega(w;\rho_n) + \|w\|^3\,\omega(s;\rho_n)].
\end{aligned}$$

This implies part (i) of the theorem.

(ii) We show the result for l_n; the proof for u_n is analogous, and consequently is omitted. Let $x \in X$. By the definition of l there exists

$t \in [a, x]$ such that $l(x) = s(x) f_1(t)/s(t)$. Let us select $y \in X_n$ such that $0 \leqslant t - y = 2\rho_n$. Since $l_n \in P_s$ on X, then

$$l_n(x) \geqslant s(x)\, l_n(y)/s(y).$$

Hence by the definition of l_n and the fact that $\theta_n \leqslant \theta$ we have

$$\begin{aligned}
l(x) - l_n(x) &\leqslant l(x) - s(x)\, l_n(y)/s(y) \\
&\leqslant s(x) f_1(t)/s(t) - \frac{s(x)}{s(y)} [f(y) - \theta_n/\theta] \\
&\leqslant \frac{s(x)}{s(t)} [f(t) - \theta/w(t)] - \frac{s(x)}{s(y)} [f(y) - \theta/w(y)] \\
&= \frac{s(x)}{s(t)\, s(y)} \{s(y)[f(t) - f(y)] + f(y)[s(y) - s(t)]\} \\
&\quad + \frac{\theta s(x)}{w(t)\, s(t)\, w(y)\, s(y)} \{w(t)[s(t) - s(y)] + s(y)[w(t) - w(y)]\} \\
&\leqslant \frac{\|s\|^2}{\tau^2} \omega(f; 2\rho_n) + \frac{\|s\|\,\|f\|}{\tau^2} \omega(s; 2\rho_n) + \frac{\theta\,\|s\|\,\|w\|}{\tau^2 \delta^2} \omega(s; 2\rho_n) \\
&\quad + \frac{\theta\,\|s\|^2}{\tau^2 \delta^2} \omega(w; 2\rho_n).
\end{aligned}$$

From this and the fact that $\omega(h; 2\rho_n) \leqslant 2\omega(h; \rho_n)$ we conclude that

$$l(x) - l_n(x) \leqslant c_4 \omega(f; \rho_n) + c_5 \omega(w; \rho_n) + c_6 \omega(s; \rho_n) \tag{4.4}$$

for all $x \in X$. On the other hand, let $x \in X$ and $t \in X_n$ be so chosen that $0 \leqslant t - x \leqslant 2\rho_n$ and $(x, t) \cap X_n = \varnothing$. Then, by the definition of $l_n \in P_s(X)$ there exists $y \in X_n \cap [a, t]$ such that

$$l_n(x) \leqslant s(x)\, l_n(t)/s(t) = s(x) f_{1,n}(y)/s(y).$$

Hence we have either

$$\begin{aligned}
l_n(x) - l(x) &\leqslant \frac{s(x)}{s(y)} [f(y) - \theta_n/w(y)] - \frac{s(x)}{s(y)} [f(y) - \theta/w(y)] \\
&= \frac{s(x)}{s(y)\, w(y)} (\theta - \theta_n), \qquad \text{if} \quad y \leqslant x,
\end{aligned}$$

or

$$
\begin{aligned}
l_n(x) - l(x) &\leqslant \frac{s(x)}{s(t)} [f(t) - \theta_n/w(t)] - [f(x) - \theta/w(x)] \\
&= \frac{s(x)}{s(t)\, w(x)} (\theta - \theta_n) \\
&\quad + \frac{s(x)[f(t) - f(x)] + f(x)[s(x) - s(t)]}{s(t)} + \theta \frac{s(t) - s(x)}{w(t)\, s(x)} \\
&\leqslant \frac{\|s\|}{\tau\delta} (\theta - \theta_n) + \frac{2\|s\|}{\tau} \omega(f; \rho_n) + \frac{2}{\tau} (\|f\| + \theta/\delta)\, \omega(s; \rho_n), \qquad \text{if} \quad y = t.
\end{aligned}
$$

Hence by using the estimation for $\theta - \theta_n$ given in part (i) of the theorem we conclude that

$$l_n(x) - l(x) \leqslant c_4 \omega(f; \rho_n) + c_5 \omega(w; \rho_n) + c_6 \omega(s; \rho_n)$$

for all $x \in X$. This and (4.4) finish the proof of part (ii).

(iii) Setting $\theta = \theta_n$ in the proof of (ii) we obtain at once the required estimation. The inequalities $\tilde{l}_n \leqslant l \leqslant u \leqslant \tilde{u}_n$ follow directly from the definitions of $l, u, \tilde{l}_n$ and $\tilde{u}_n$.

(iv) This follows immediately from the definitions of l_n, u_n, $\tilde{l}_n$ and $\tilde{u}_n$. ∎

5. Timan's Approximation Problem and Measures of Noncompactness

Let X be a set and ρ be a semi-metric on X, i.e., a real-valued function defined on $X \times X$ and satisfying the conditions: $\rho(x, y) \geqslant 0$, $\rho(x, x) = 0$, $\rho(x, y) = \rho(y, x)$ and $\rho(x, y) \leqslant \rho(x, z) + \rho(z, y)$ for all $x, y, z \in X$. If a semi-metric ρ is a bounded function on $X \times X$ then it is called a bounded semi-metric. Define

$$H_\rho = \{h \in B: |h(x) - h(y)| \leqslant \rho(x, y) \text{ for all } x, y \in X\}.$$

If G is a bounded subset of B then $K(G, \lambda)$ will denote the ball centered at G with radius λ, i.e.,

$$K(G, \lambda) = \bigcup_{g \in G} K(g, \lambda),$$

where $K(g, \lambda)$ is the ball about g with radius λ in B, i.e.,

$$K(g, \lambda) = \{h \in B: \|h - g\|_w < \lambda\}.$$

Also, define

$$\alpha(G) = \inf\{\varepsilon > 0: G \text{ can be covered by a finite number of sets of diameter } \leqslant \varepsilon\}$$

and

$$\mu(G) = \inf\{\varepsilon > 0: G \text{ can be covered by a finite number of balls of radius } \leqslant \varepsilon\}.$$

The set-functions α and μ are called Kuratowski and ball's measures of noncompactness (see [10] and [5, 6, 11, 12]), respectively.

LEMMA 5.1. *Let ρ be a bounded semi-metric. Then the set H_ρ is admissible with respect to each $r \in B$ and the functions l and u from Definition* 2.1 *are equal to*

$$l(x) = \sup_{z \in X} [r(z) - \rho(x, z)] \qquad \textit{and} \qquad u(x) = \inf_{z \in X} [r(z) + \rho(x, z)]$$

for all $x \in X$.

Proof. Condition (iii) in Definition 2.1 is trivial for H_ρ. We verify only condition (i) in Definition 2.1; condition (ii) can be shown in the same way. First, by a boundness of ρ and r it follows that $l \in B$. Clearly, $l(x) \geqslant r(x) - \rho(x, x) = r(x)$. Thus $l \geqslant r$. In addition, by the triangle inequality for semi-metric ρ we have

$$r(z) - \rho(x, z) \leqslant r(z) - \rho(y, z) + \rho(x, y)$$

for each $z \in X$. Taking the supremum over z of the right-hand side and the supremum over z of the left-hand side of the last inequality, we conclude that

$$l(x) \leqslant l(y) + \rho(x, y). \tag{5.1}$$

Conversely, $r(z) - \rho(y, z) \leqslant r(z) - \rho(x, z) + \rho(x, y)$. Hence $l(y) \leqslant l(x) + \rho(x, y)$. This and (5.1) imply that $l \in H_\rho$. Therefore, it remains to prove that $g \geqslant l$, while $g \geqslant r$ and $g \in H_\rho$. To this purpose, suppose that

$$g(y) \geqslant r(y) \qquad \text{and} \qquad g(y) \leqslant g(x) + \rho(x, y)$$

for all $x, y \in X$. Then

$$g(x) \geqslant g(y) - \rho(x, y) \geqslant r(y) - \rho(x, y)$$

for all $x, y \in X$. Taking supremum over y of the right-hand side, we infer that $g \geqslant l$. ∎

THEOREM 5.2. *Let ρ be a bounded semi-metric, $f \in B \backslash H_\rho$, and θ, l and u be defined by*

$$\theta = \sup_{x, y \in X} d(x, y), \qquad l(x) = \sup_{z \in X} [f_1(z) - \rho(x, z)]$$

and

$$u(x) = \inf_{z \in X} [f_2(z) + \rho(x, z)],$$

where

$$d(x, y) = \frac{w(x)\, w(y)}{w(x) + w(y)} [f(x) - f(y) - \rho(x, y)],$$

$$f_1 = f - \theta/w \qquad \textit{and} \qquad f_2 = f + \theta/w.$$

Then, $l, u \in H_\rho$, $l \leqslant u$, the set of all best weighted approximations to f in H_ρ is equal to $[l, u] \cap H_\rho$, and the error $\theta_f = \inf_{g \in H_\rho} \|f - g\|_w$ is equal to θ. Moreover, the error-determining set N_f is equal to

$$N_f = (Z_{f_1 - l} \cap Z_{f_1 - u}) \cup (Z_{f_2 - l} \cap Z_{f_2 - u}).$$

Proof. By Theorem 2.2 and Lemma 5.1 it remains to show that $\theta_f = \theta$. Suppose that $g \in H_\rho$. Then $g(x) \leqslant g(y) + \rho(x, y)$ and

$$\begin{aligned} f(x) - f(y) - \rho(x, y) &\leqslant f(x) - f(y) - g(x) + g(y) \\ &\leqslant \|f - g\|_w [1/w(x) + 1/w(y)] \end{aligned}$$

for all $x, y \in X$. Hence $\|f - g\|_w \geqslant d(x, y)$ for all $x, y \in X$. Thus $\theta_f \geqslant \theta$. For the converse inequality, we show that $\|f - l\|_w \leqslant \theta$, where $l \in H_\rho$ is defined above. First, we note that

$$w(x)[f(x) - l(x)] \leqslant w(x)[f(x) - f_1(x) + \rho(x, x)] = \theta \tag{5.2}$$

for all $x \in X$. Conversely, for each $\varepsilon > 0$ and $x \in X$ there exists $t \in X$ such that

$$l(x) \leqslant f_1(t) - \rho(x, t) - f(x) + \varepsilon.$$

Hence

$$\begin{aligned} w(x)[l(x)-f(x)] &\leqslant w(x)[f(t)-f(x)-\rho(x,t)-\theta/w(t)+\varepsilon] \\ &\leqslant w(x)\Bigg[f(t)-f(x)-\rho(x,t) \\ &\quad -\frac{w(x)}{w(t)+w(x)}(f(t)-f(x)-\rho(t,x))+\varepsilon\Bigg] \\ &= d(t,x)+\varepsilon w(x) \leqslant \theta+\varepsilon \|w\|. \end{aligned}$$

Since ε is arbitrary and independent on x, we then conclude that $w(x)[l(x)-f(x)] \leqslant \theta$ for all $x \in X$. Combining this inequality with (5.2), we obtain $\|f-l\|_w \leqslant \theta$. Hence $\theta_f \leqslant \theta$ and the theorem is proved. ∎

COROLLARY 5.3. *Let ρ be a bounded metric, $w \in C_b = C_b(X_\rho)$, $G_\rho = H_\rho \cap C_b$ and $f \in C_b \backslash G_\rho$. Then, in addition to the results of Theorem* 5.2 *we have, $l, u \in G_\rho$ and*

$$\theta_f = \inf_{g \in H_\rho} \|f-g\|_w = \inf_{g \in G_\rho} \|f-g\|_w.$$

Proof. The functions $f_1(x)-\rho(x,y)$ and $f_2(x)+\rho(x,y)$ are continuous functions of two variables $x, y \in X$. Hence by Theorem 5.2 the corollary follows. ∎

For the remainder of this section it is assumed that the set H_ρ consists of all h functions not necessarily bounded on X such that

$$|h(x)-h(y)| \leqslant \rho(x,y) \qquad \text{for all} \quad x, y \in X.$$

Then, reviewing the proof of Theorem 5.2 we can easily derive the following:

COROLLARY 5.4. *The formula for θ_f given in Theorem* 5.2 *is true for any semi-metric ρ not necessarily bounded.*

THEOREM 5.5. *Let ρ, σ be two semi-metrics.* Then the *deviation of H_σ from H_ρ is equal to*

$$\theta(H_\rho, H_\sigma) = \sup_{x,y \in X} e(x,y),$$

where

$$e(x,y) = \frac{w(x)\,w(y)}{w(x)+w(y)}\,[\rho(x,y)-\sigma(x,y)].$$

Proof. By (2.3), Theorem 5.2 and Corollary 5.4 we have

$$\theta(H_\rho, H_\sigma) = \sup_{f \in H_\rho} \sup_{x, y \in X} d(x, y), \tag{5.3}$$

where d is defined in Theorem 5.2. Now, let f be a given function in H_ρ. Then $f(x) - f(y) \leqslant \rho(x, y)$ and $f(x) - f(y) - \sigma(x, y) \leqslant \rho(x, y) - \sigma(x, y)$ for all $x, y \in X$. Multiplying this inequality by $w(x)\, w(y)/(w(x) + w(y))$, taking the supremum of the right-hand side over x, y and the supremum of the left-hand side over $x, y \in X$ and $f \in H_\rho$, and using of (5.3), we obtain

$$\theta(H_\rho, H_\sigma) \leqslant \sup_{x, y \in X} e(x, y). \tag{5.4}$$

Conversely, let $x_0, y_0 \in X$ be such that

$$\sup_{x, y \in X} e(x, y) \leqslant e(x_0, y_0) + \varepsilon, \tag{5.5}$$

where $\varepsilon > 0$ is arbitrary. Put $h(x) = \rho(x, y_0)$. Since $-\rho(x, y) \leqslant \rho(x, y_0) - \rho(y, y_0) = h(x) - h(y) \leqslant \rho(x, y)$, then $h \in H_\rho$. Hence by (5.3) and (5.5) we obtain

$$\begin{aligned} \theta(H_\rho, H_\sigma) &= \frac{w(x_0)\, w(y_0)}{w(x_0) + w(y_0)} \left[h(x_0) - h(y_0) - \sigma(x_0, y_0)\right] = e(x_0, y_0) \\ &\geqslant \sup_{x, y \in X} e(x, y) - \varepsilon. \end{aligned}$$

Since ε is arbitrary we then have

$$\theta(H_\rho, H_\sigma) \geqslant \sup_{x, y \in X} e(x, y).$$

Combining this inequality with (5.4) we obtain the result. ∎

We note that formulae for θ_f and $\theta(H_\rho, H_\sigma)$ from Theorems 5.2 and 5.5 were first proven in a different way by Timan in [14] for the case $w \equiv 1$.

THEOREM 5.6. *Let X be a compact metric space with a metric ρ and let G be a bounded subset of $C(X)$. Then*

$$\mu(G) = \inf_{\lambda > 0} \sup_{g \in G} \sup_{\substack{x, y \in X \\ \rho(x, y) \leqslant \lambda}} \frac{w(x)\, w(y)}{w(x) + w(y)} |g(x) - g(y)|.$$

Proof. Let $g \in G$ and $\varepsilon > 0$ be arbitrarily fixed. First, by the definition of

μ there exists a finite set $H \subset C(X)$ $(\text{card}(H) < \infty)$ such that $G \subset K(H, \mu(G) + \varepsilon)$. Hence

$$\begin{aligned} |g(x) - g(y)| &\leqslant |g(x) - h(x)| + |h(x) - h(y)| + |h(y) - g(y)| \\ &\leqslant [1/w(x) + 1/w(y)](\mu(G) + \varepsilon) + |h(x) - h(y)| \end{aligned}$$

for all $h \in H$ and all $x, y \in X$. It follows that

$$\frac{w(x)\, w(y)}{w(x) + w(y)} |g(x) - g(y)| \leqslant \mu(G) + \varepsilon + \frac{\|w\|^2}{2\delta} \sup_{h \in H} \omega(h; \lambda) \tag{5.6}$$

for all $x, y \in X$ such that $\rho(x, y) \leqslant \lambda$. Note that by the finiteness of $H \subset C(X)$ it follows that

$$\inf_{\lambda > 0} \sup_{h \in H} \omega(h; \lambda) = 0.$$

Consequently, taking the supremum first over $x, y \in X$ $(\rho(x, y) \leqslant \lambda)$ and $g \in G$, the infimum over $\lambda > 0$ of the left-hand side of inequality (5.6) and next the infimum over $\lambda > 0$ of the right-hand side of (5.6), we obtain

$$a(G) := \inf_{\lambda > 0} \sup_{g \in G} \sup_{\substack{x, y \in X \\ \rho(x, y) \leqslant \lambda}} \frac{w(x)\, w(y)}{w(x) + w(y)} |g(x) - g(y)| \leqslant \mu(G) + \varepsilon. \tag{5.7}$$

Now, define two metrics ρ_1 and ρ_2 on X by

$$\begin{aligned} \rho_1(x, y) &= 0, \qquad \text{if} \quad x = y \\ &= \rho(x, y) + |g(x) - g(y)| + a(G)/\beta, \qquad \text{otherwise} \end{aligned}$$

and

$$\begin{aligned} \rho_2(x, y) &= 0, \qquad \text{if} \quad x = y \\ &= \rho(x, y) + \sup_{g \in G} |g(x) - g(y)|, \qquad \text{otherwise,} \end{aligned}$$

where

$$\beta = \sup_{x, y \in X} \frac{w(x)\, w(y)}{w(x) + w(y)}.$$

Denote

$$H_i = \{f \in C(X_{\rho_2}) : |f(x) - f(y)| \leqslant \rho_i(x, y) \text{ for all } x, y \in X\}, \qquad i = 1, 2.$$

Then by the definitions of H_1 and ρ_1 we conclude that $g \in H_1$. Hence by Theorem 5.5 we obtain

$$\theta(g, H_2) \leqslant \theta(H_1, H_2) = \sup_{x,y \in X} \frac{w(x)\, w(y)}{w(x) + w(y)} [\rho_1(x, y) - \rho_2(x, y)]$$

$$\leqslant \frac{a(G)}{\beta} \sup_{x,y \in X} \frac{w(x)\, w(y)}{w(x) + w(y)} = a(G).$$

Since by the Arzelá theorem the subset H_2 of $C(X_{\rho_2})$ is locally compact in $C(X_{\rho_2})$ then for each $g \in G$ there exists $h \in H_2$ such that

$$\theta(g, H_2) = \inf_{r \in H_2} \| g - r \|_w = \| g - h \|_w \leqslant a(G). \tag{5.8}$$

Denote the bounded set of all such $h \in H_2$ by H_3 ($\operatorname{diam}(H_3) \leqslant 2a(G)$). Then by (5.8) we have $G \subset K(H, a(G) + \varepsilon)$, where $H = \{h_1, ..., h_k\}$ is the ε-net of the set H_3. Since $\varepsilon > 0$ is arbitrary, then from the last inclusion, the definition of μ and (5.7) it follows that the proof is completed. ∎

Finally, we note that between Kuratowski and ball's measures of noncompactness α and μ the relation

$$\mu(G) \leqslant \alpha(G) \leqslant 2\mu(G)$$

holds. Therefore, the formula for μ, given in Theorem 5.6, provides a useful, lower and upper bounds for Kuratowski's measure of noncompactness in $C(X)$, where X is a compact metric space.

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